On the Decentralized Generation of the RSA Moduli in Multi-Party Settings

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Abstract

RSA cryptography is still widely used. Some of its applications (e.g., distributed signature schemes, cryptosystems) do not allow the RSA modulus to be generated by a centralized trusted entity. Instead, the factorization must remain unknown to all the network participants. To this date, the existing algorithms are either computationally expensive, or limited to two-party settings. In this work, we design a decentralized multi-party computation algorithm able to generate efficiently the RSA modulus.

1 Introduction

Rivest-Shamir-Adleman (RSA) [14] is one of the first public key encryption systems, and it is still widely used. RSA uses the product of two large prime numbers p and q to compute a public modulus N. The security of RSA is guaranteed by the difficulty of the integer factorization of the public modulus: its factorization would provide the prime numbers p and q and, thus, the corresponding private key.

The simplest technique to generate RSA keys is by using an agent trusted by the parties who does not disclose the factorization. However, many applications using RSA (e.g., distributed signature schemes [12], threshold cryptosystems [11], multi party computations [8]) require to generate the RSA keys in a distributed way. In other contexts, for instance when using verifiable delay functions [4] for spam prevention in distributed ledger technologies [7], the private key does not have to be computed, and the algorithm require the decentralized generation of the public modulus only.

In this document, we propose a multi-party protocol generating a public RSA modulus, which extends the state-of-the-art two-party algorithm discussed in [Frederiksen].

The rest of the paper is organized as follows: in Section 2, we present the previous work on RSA key generation, and in Section 3 we review in detail the fastest algorithm to date [Frederiksen]; then, in Section 4, we describe our algorithm working in a multi-party setting; after that, we conclude our

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paper with a discussion about network overhead and algorithm complexity in Section 5.

2 State of the art

Fiat and Shamir seminal paper on signature schemes [10] has paved the way to the research on usnig public modulus with unknown factorization. Later, Boneh and Franklin designed a distributed modulus generation algorithm including a biprimality testing algorithm in a multi-party setting [5]. The algorithm works with semi-honest adversaries¹ in a honest majority.

Another direction was taken by Algesheimer [1] and by Damgard and Mikkelsen [9]. The authors suggested to compute a full primality test of the factors while keeping them secret instead of a biprimality test by executing a distributed Miller-Rabin test [13]. Gilbo presented a secure version of the protocol based on the Boneh and Franklin technique [5].

The first work on two-parties protocol with one of the party being semihonest has began with [6] but was found to be insecure by [3]

More recently, Frederiksen [Frederiksen] proposed an algorithm using a new distributed product routine running in a two-party setting both in a semi-honest and in a malicious environment by using modern techniques such as Oblivious Transfer extensions [2]; this solution runs in 40 seconds, compared to 15 minutes of the previous fastest protocol. In this paper, we propose a multi party algorithm for multi-party RSA modulus generation by extending the Frederiksen's protocol [Frederiksen] to more than two parties. We start by reviewing the original protocol in the next section.

3 Frederiksen's protocol

3.1 Algorithm set up

3.1.1 Objective

The algorithm aims to generate two large *prime* random numbers p and q of the same order (i.e., having the same bit-length) and the distributed computation of their product $N = p \cdot q$. The numbers p and q are k-bit numbers, i.e., $p, q \in [1, 2^k]$. A fundamental requirement is that none of the party involved in the computation can be able to retrieve either p or q.

3.1.2 System model

Consider a network made of a set \mathcal{N} of two nodes $\mathbf{P}_1, \mathbf{P}_2 \in \mathcal{N}$ which decide to collaborate to run the protocol. The protocol makes the following additional assumptions:

A.1 Semi-honest participants. Both nodes follow the protocol. However, one of them could try to guess the secret shares of the other one to retrieve the factorization of the public modulus.

¹Semi-honest nodes precisely follow the protocol (like honest nodes) but try to collect information for malicious purposes during their communications. They are also called *honest but curious*.

A.2 Reliable networking layer. Every message is received in the same way it has been sent with probability one and in finite time.

3.1.3 Protocol outline

At a high level, the proposed algorithm involves the following steps:

- P.1 Random number generation. The two active nodes \mathbf{P}_1 and \mathbf{P}_2 respectively generate the numbers p_1 , q_1 and p_2 , q_2 .
- P.2 Fast trial divisions. The active nodes run fast trial divisions by small primes numbers on both $p = p_1 + p_2$ and $q = q_1 + q_2$ to quickly discard wrong candidates. If p or q fail the test, the algorithm restarts from P.1.
- P.3 RSA modulus computation. The two parties compute $N = (p_1 + p_2) \cdot (q_1 + q_2)$ in a distributed way without revealing information on key's parts.
- P.4 Biprimality test. A biprimality test verifies whether N is a product of two prime numbers whp. If the biprimality test fails, the algorithm restarts from P.1.

In the next subsection, we will describe rigorously each part of the protocol.

3.2 Detailed protocol

3.2.1 Random number generation

The goal of this step, is for node \mathbf{P}_1 (resp. \mathbf{P}_2) to pick two numbers $p_1, q_1 \in [0, 2^k - 1]$ (resp. $p_2, q_2 \in [0, 2^k - 1]$). Let $p \triangleq p_1 + p_2$ and $q \triangleq q_1 + q_2$ be the sum of the number generated by the two nodes. However the biprimality test (step P.4 of the protocol) requires

$$p \equiv 3(\bmod 4) \land q \equiv 3(\bmod 4). \tag{1}$$

Hence, the protocol enforces that \mathbf{P}_1 (resp. \mathbf{P}_2) randomly picks two numbers $\hat{p}_1, \, \hat{q}_1 \in [1, 2^{k-2}]$ (resp. $\hat{p}_2, \, \hat{q}_2 \in [1, 2^{k-2}]$). After that, node \mathbf{P}_1 (resp. node \mathbf{P}_2) concatenates two zeros (resp. two ones) to satisfy Eq. (1), i.e., $p_1 \triangleq \hat{p}_1 || \{0; 0\}$ and $q_1 \triangleq \hat{q}_1 || \{0; 0\}$ (resp. $p_2 \triangleq \hat{p}_2 || \{1; 1\}$ and $q_2 \triangleq \hat{q}_2 || \{1; 1\}$).

3.2.2 Fast trial divisions

As a protocol optimization, we can discard some trivial non-prime candidates to reduce the number of the algorithm iterations. The idea is to divide the candidates with small prime numbers.

Let $B \in \mathbb{N}$ be a certain threshold, which indicates the number of prime numbers that we want to test. For each prime $\beta < B$, \mathbf{P}_1 and \mathbf{P}_2 run a β -divisibility test for p and q. The β -divisibility test consists in computing the remainder of $(p_1 + p_2) \pmod{\beta}$ while keeping secret p_1 and p_2 . The above is described in algorithm 1. The parties test if $p_1 + p_2 \equiv 0 \pmod{\beta}$ or, equivalently, $p_1 \equiv -p2 \pmod{\beta}$. We use the 1-out-of- β Oblivious Transfer (OT) algorithm² to hide the rest of the modulus β of p_2 to \mathbf{P}_1 . The same test has to be ran for q.

 $^{^2\}mathrm{We}$ refer the interested reader to Appendix 5.4 for further information on the algorithm.

In the algorithm, \mathbf{P}_1 inputs $\{r_0,\ldots,r_{\beta-1}\}\in[0,2^k-1]^\beta$ to the OT, while \mathbf{P}_2 requests $r_{-p_2\pmod{\beta}}$ from the OT and send it to \mathbf{P}_1 . Then, \mathbf{P}_1 can check if $r_{-p_2\pmod{\beta}}=r_{p_1\pmod{\beta}}$: if yes, then $p1\equiv -p_2\pmod{\beta}$ and the protocol has to be restarted.

Finally, one can note that this step is parallelizable by running different β -divisibility tests simultaneously with different values of β .

Algorithm 1 β -divisibilty-test

 \mathbf{P}_1 and \mathbf{P}_2 initialize a 1-out-of- β OT

 $\mathbf{P}_1 \text{ generates } \{r_0, \dots, r_{\beta-1}\} \in [0, 2^k - 1]^{\beta}$

 \mathbf{P}_1 inputs $\{r_0,\ldots,r_{\beta-1}\}$ to the OT

 \mathbf{P}_2 computes $a = -p_2 \mod \beta$

 \mathbf{P}_2 receives r_a from the OT

 \mathbf{P}_1 computes $b = p_1 \mod \beta$

 \mathbf{P}_1 sends r_b to \mathbf{P}_2

if $r_a = r_b$ then \mathbf{P}_2 sends \perp else \mathbf{P}_2 sends \top

3.2.3 RSA modulus computation

This step aims to compute the product between p and q, i.e.,

$$N = p \cdot q$$

$$= (p_1 + p_2) \cdot (q_1 + q_2)$$

$$= p_1 \cdot q_1 + p_1 \cdot q_2 + p_2 \cdot q_1 + p_2 \cdot q_2.$$

While the terms $p_1 \cdot q_1$ and $p_2 \cdot q_2$ can be computed locally by \mathbf{P}_1 and \mathbf{P}_2 , the two remaining terms must be computed in a distributed way. To this end, the protocol defines the procedure distr_product.

Let us assume \mathbf{P}_1 knows $a \in \mathbb{N}$ and \mathbf{P}_2 knows $b \in \mathbb{N}$, and the two nodes want to compute $a \cdot b$ without disclosing their number to the other party. Moreover, let

$$l \triangleq \max(|\log_2(a)| + 1, |\log_2(b)| + 1)$$

be the minimum number of bits necessary to represent a and b. Then, given an input $x \in \mathbb{N}$, for bit i, we can define the function

$$\mathcal{B}_i(x) = \begin{cases} 0, & \text{if the i-th bit of } x \text{ is } 0; \\ 1, & \text{otherwise.} \end{cases}$$

Note that we count bits from the least significant digit.

For each bit i of x, \mathbf{P}_1 generates

$$c_i^1 = c_i^0 + a$$
, where $tc_i^0 \in [0, 2^l - 1]$,

and the instantiates an OT with $\{c_i^0,c_i^1\}$ as an input. Then, \mathbf{P}_2 requests $c_i^{\mathcal{B}_i(b)}=c_i^0+a\cdot\mathcal{B}_i(b)$. Finally, \mathbf{P}_1 computes

$$s_1 = -\sum_{i=0}^{l-1} c_i^0 \cdot 2^i,$$

and P_2 computes

$$s_2 = \sum_{i=0}^{l-1} c_i^{\mathcal{B}_i(b)} \cdot 2^i.$$

Hence.

$$s_1 + s_2 = \sum_{i=0}^{l-1} \mathcal{B}_i(b) \cdot a \cdot 2^i = a \cdot b.$$

To calculate $p_1 \cdot q_2 + q_1 \cdot p_2$ without information leakage, \mathbf{P}_1 will compute s_1^p (resp. s_1^q) and \mathbf{P}_2 will compute s_2^q (resp. s_2^p) by running distr_product(p_1 , q_2) (resp. distr_product (q_1, p_2)). Finally, \mathbf{P}_1 will share $s_1^p + s_1^q$ with \mathbf{P}_2 and \mathbf{P}_2 will share $s_2^p + s_2^q$ such as

$$s_1^p + s_2^p + s_1^q + s_2^q = p \cdot q.$$

Importantly, the algorithm allows each party to only have partial information of the output, while their sum is the exact output. In order to dissimulate information, the parties only share the sum of all the partial product they compute.

Algorithm 2 distr_product

We have \mathbf{P}_1 which knows $a = \sum a_i 2^i$ and \mathbf{P}_2 which knows $b = \sum b_i 2^i$ with $i \in \{0; l-1\}$ and $a_i, b_i \in \{0, 1\}$

 $\mathbf{P}_1 \text{ sets } s_1 = 0$

 $\mathbf{P}_2 \text{ sets } s_2 = 0$

for i = 0 to i = l - 1

 \mathbf{P}_1 and \mathbf{P}_2 initialize a 1-out-of-2 OT

 $\mathbf{P}_1 \text{ generates } c_0 \in \{0, 2^l - 1\}$

 \mathbf{P}_1 computes $c_1 = c_0 + a \mod 2^l$

 \mathbf{P}_1 inputs $\{c_0, c_1\}$ into the OT

 \mathbf{P}_2 receives c_{b_i} from the OT

 $\mathbf{P}_1 \text{ computes } s_1 = s_1 - c_0 2^i$

 P_2 computes $s_2 = s_2 + c_{b_i} 2^i$

Both parties keep s_1 and s_2 secret for future usage

Algorithm 3 Distributed computation of $p \cdot q$

P1 and **P2** compute respectively α_1^2 and β_2^1 via distr_product

 ${f P1}$ and ${f P2}$ compute respectively eta_1^2 and $lpha_2^1$ via distr_product of p_2q_1

P1 computes $f_1 = p_1 q_1 + \alpha_1^2 + \beta_1^2$ **P2** computes $f_2 = p_2 q_2 + \alpha_2^1 + \beta_2^1$

P1 and **P2** share f_1 and f_2 to compute $N = f_1 + f_2$

Biprimality test

This test, inspired by [5], is composed of two parts, and requires to perform s times a random filtering test.

Algorithm 4 Distributed filtering biprimality test

P1 samples $\gamma \in \mathbb{Z}_N^{\times}$ with Jacobi symbol over N equal to 1

P1 sends γ to **P2**

P1 computes $\gamma_1 = \gamma^{\frac{N+1-p_1-q_1}{4}} \mod N$ and sends it to P2 P2 computes $\gamma_2 = \gamma_1 \cdot \gamma^{\frac{-p_2-q_2}{4}} \mod N$

if $\gamma_2 = \pm 1 \; \mathbf{P2} \; \mathrm{sends} \; \top \; \mathbf{else} \; \mathbf{P2} \; \mathrm{sends} \; \bot$

Then the parties verify in a distributed way that gcd(N, p + q - 1) = 1.

Algorithm 5 Distributed test of gcd(N, p + 1 - 1)

 $\overline{\mathbf{P1}}$ and $\overline{\mathbf{P2}}$ generate respectively r_1 and r_2 in \mathbb{Z}_N

They compute shares α_1 , α_2 of $r_1(p_2 + q_2 - 1) \mod 2^{3k_2}$

They compute shares β_1 , β_2 of $r_2(p_1+q_1) \mod 2^{3k_2}$

P1 sends $s_1 = r_1(p_1 + q_1) + \alpha_1 + \beta_1 \mod N$ to **P2**

P2 sends $s_2 = r_2(p_2 + q_2 - 1) + \alpha_2 + \beta_2 \mod N$ to **P1**

Both parties check that $gdc(s_1 + s_2, N) = 1$ and return \top otherwise return ⊥

4 Generalization to n participants

The aforementioned two-party protocol is proven to work securely in a semihonest environment. Here, we will present our generalization to n parties while keeping the same high-level structure. Also, we make sure that the computational complexity and the network overhead are reasonably low, to keep the efficiency of the algorithm similar to the one in [Frederiksen].

System model

We consider a set \mathcal{N} of $n \in \mathcal{N}$ nodes which participates to the protocol. We refer to node i as P_i . For the sake of simplicity, we assume the number of participants to be $n = 2^t$, where $t \in \mathbb{N}$.

We keep the assumption of a perfect networking layer and semi-honest participants. However, we also consider the scenario where some of them can collude and communicate their private information in order to derive the private key. From the point of view of an honest node, it is not possible to detect such a behaviour as all nodes actually follow the protocol.

Key's parts generation

Each node \mathbf{P}_i has to generate a pair $p_i,q_i\in[1,2^k]$. Moreover, we define $p\triangleq$ $\sum_{i=1}^{n} p_i$ and $q \triangleq \sum_{i=1}^{n} q_i$. Since, the biprimality test requires that $p \equiv 3 \pmod{4}$ and $q \equiv 3 \pmod{4}$ (see Eq. (1)), the protocol enforces that:

• \mathbf{P}_1 randomly picks two numbers $\hat{p}_1, \, \hat{q}_1 \in [1, 2^{k-2}]$ and concatenates two ones to satisfy Eq. (1), i.e., $p_1 \triangleq \hat{p}_1 ||\{1;1\}|$ and $q_1 \triangleq \hat{q}_1 ||\{1;1\}|$

• all the other nodes i, where $i \in \mathcal{N} - \{1\}$, randomly picks two numbers \hat{p}_i , $\hat{q}_i \in [1, 2^{k-2}]$, and concatenates two zeros, i.e., $p_i \triangleq \hat{p}_i || \{0; 0\}$ and $q_i \triangleq \hat{q}_i || \{0; 0\}$.

Node \mathbf{P}_1 can be selected through leader election (potentially based on some node characteristics, such as reputation or hash). We note that the selection of this node does not influence the outcome of the protocol, as it only affects the generation of the two last bits of its secret numbers.

4.3 Fast trial divisions

Generalizing the β -division to n parties is complicated as the original algorithm was specifically designed for two parties. We unstress the constraint of using OTs to communicate the secret shares remainder modulus β . Our β -division works in t turns. We set $\mathcal{N}_0 = \mathcal{N}$ and during the first turn, there is a consensus on a selection of a subset $\mathcal{N}_1 \subset \mathcal{N}$ and a bijection

$$f_1: \mathcal{N}_0 \setminus \mathcal{N}_1 \to \mathcal{N}_1$$

between \mathcal{N}_1 and the set of non-selected nodes. Each non-selected node i will send its secret share $p_i \mod \beta$ to its associated node j in \mathcal{N}_1 which will sum it with its own secret share modulus β . In the second turn, we will again select a subset $\mathcal{N}_2 \subset \mathcal{N}_1$ of half of the nodes in \mathcal{N}_1 and a bijection

$$f_2: \mathcal{N}_1 \setminus \mathcal{N}_2 \to \mathcal{N}_2$$

between the selected and non-selected nodes such that the non-selected nodes will send the sum of the private share modulus β they know. And the algorithm will iteratively divide the selected sets until only \mathcal{N}_t which contains only one node that will be able to output if $\sum_{i\in\mathcal{N}}p_i\mod\beta==0$. The algorithm 6 shows a pseudocode of this procedure.

The major issue in this algorithm is how to construct $f_k \colon \mathcal{N}_{k-1} \setminus \mathcal{N}_k \to \mathcal{N}_k$ in a decentralized way such that everybody have the same and with a minimum amount of communications. We propose in algorithm 7 a way to solve this problem using a deterministic attribution of the associations using a hashing function preventing malicious parties to manipulate this part of the protocol. We assume the parties agree before the fast trials part on a seed and a certain hashing function H such that we can construct $H_m \colon \{0,1\}^* \to [1,m]$ for $m \in \mathbb{N}$. The way this seed and H are generated depends on the network's goals and policies but they can agree to use the SHA256 hashing function and generate the seed using a distributed number generator. This technique allows to have no communication for finding consensus during the protocol.

Algorithm 6 Generalization of the fast trial division

```
\begin{aligned} & \mathbf{P}_{i}.v \leftarrow p_{i}, \, \forall i \in \mathcal{N} \\ & \text{for } k \leftarrow 1 \text{ to } k \leftarrow t \\ & \text{The nodes in } \mathcal{N}_{k-1} \text{ select } \mathcal{N}_{k} \subset \mathcal{N}_{k-1} \text{ containing half the nodes of } \mathcal{N}_{k-1} \\ & \text{and } f_{k} \\ & \text{foreach party } \mathbf{p} \in \mathcal{N}_{k-1} \setminus \mathcal{N}_{k} \\ & \mathbf{p} \text{ sends } \mathbf{p}.v \mod \beta \text{ to } f_{k}(\mathbf{p}) \\ & f_{k}(\mathbf{p}) \text{ computes } f_{k}(\mathbf{p}).v \leftarrow (f_{k}(\mathbf{p}.v) + \mathbf{p}.v) \mod \beta \end{aligned}
The last party \mathbf{p}_{final} in \mathcal{N}_{t} sends \bot if \mathbf{p}_{final}.v = 0 else \top
```

Algorithm 7 Construction of f_k

```
\overline{m \leftarrow 2^{t-k-1}}
We assume \mathcal{N}_{k-1} \setminus \mathcal{N}_k and \mathcal{N}_k are indexable data structures of m elements
D \text{ is a data structure mapping } \mathcal{N}_{k-1} \setminus \mathcal{N}_k \to \mathcal{N}_k
A \leftarrow []
\text{for } i \leftarrow 1 \text{ to } i \leftarrow m
j \leftarrow 0
\text{do}
c \leftarrow H_m(\beta + \text{seed} + "|" + k + "|" + j)
\text{while } c \in A
D[\mathcal{N}_{k-1} \setminus \mathcal{N}_k[i]] \leftarrow \mathcal{N}_k[c]
return D
```

The input of the hashing function in this algorithm contains β , \mathtt{seed},k and j.

- β is here to change the output of the function at each division trial to prevent people from getting information from the same parties each time without having to agree on a new seed each time.
- k plays the same role be between each turn of one β -division
- j is here to make sure that A is a permutation of [1, m] and does not contain twice the same number while allowing people to compute on their own without communicating.

4.4 Distributed multiplication

In this subsection, we want to compute $\sum_i p_i \cdot \sum_i q_i$ in a similar way to the Frederiksen's algorithm. First, note that

$$\sum_{i=1}^{n} p_i \cdot \sum_{i=1}^{n} q_i = \sum_{i=1}^{n} (p_i \cdot q_i) + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (p_i \cdot q_j + p_j \cdot q_i)$$
 (2)

The first term of the left hand side of Equation 2 can be computed solely by every party. As for the second term, for each pair $i, j \in \mathcal{N}^2$ with i < j,

the nodes \mathbf{P}_i and \mathbf{P}_j have to compute $p_i \cdot q_j + p_j \cdot q_i$ in a distributed way without disclosing their secret shares. We then suggest that \mathbf{P}_i and \mathbf{P}_j run the distr_product routine and compute $x_i^{j,p}$ for \mathbf{P}_i and $x_j^{i,q}$ for \mathbf{P}_i for computing $p_i \cdot q_j$ and respectively they would compute $x_i^{j,q}$ and $x_j^{i,p}$ while computing $p_j \cdot q_i$. Finally, we have for each party \mathbf{P}_i , \mathbf{P}_i computes

$$f_i = p_i \cdot q_i + \sum_{j \in \mathcal{N}, j \neq i} x_i^{j,p} + x_i^{j,q},$$

and broadcast it to everyone so that everybody can compute $N = \sum_{i=1}^{n} f_i$. At that point, N is a public information to everyone.

Algorithm 8 Generalized distributed multiplication

```
\begin{aligned} f_i &\leftarrow 0 \text{ for } i \in \mathcal{N} \\ \text{ for } 1 &\leq i \leq j \leq n \\ & \text{ if } i = j \\ & f_i += p_i q_i \\ \text{ else} \end{aligned} We compute p_i q_j and p_j q_i as in algorithm 9 f_i &+= x_i^{j,p} + x_i^{j,q} \\ f_j &+= x_j^{j,p} + x_j^{j,q} \end{aligned} The parties share f_i \ \forall i \mathcal{N} to compute N \leftarrow \sum_{i \in \mathcal{N}} f_i
```

Algorithm 9 Computing $p_i q_j + p_j q_i$

We have two parties, $\mathbf{P}i$ and $\mathbf{P}j$ with i < j

 $\mathbf{P}i$ and $\mathbf{P}j$ compute respectively $x_i^{j,p}$ and $x_j^{i,q}$ via distributed computation of p_iq_i

 $\mathbf{P}i$ and $\mathbf{P}j$ compute respectively $x_i^{j,q}$ and $x_j^{i,p}$ via distributed computation of p_jq_i

4.5 Biprimality test

First part

In this section we keep the same idea of testing for a certain γ randomly generated if $\gamma^{N+1-p-q} = \pm 1$. This part can be easily done in a decentralized way with properties offered by the exponentiation. The parties elect a leader \mathbf{P}_e that will generate a number $\gamma \in \mathbb{Z}_N^{\times}$ with Jacobi symbol over N equal to 1 and shares it with everybody. Then each party $\mathbf{P}_i, i \in \mathcal{N}$ will compute $\gamma_i = \gamma^{-p_i-q_i}$ and send it to \mathbf{P}_e . Once \mathbf{P}_e received all the gamma_i , it will compute $\operatorname{gammafinal} = \gamma^{N+1} \cdot \prod_{i=1}^n \gamma_i \mod N$. Finally, if $|\gamma_{final}| = 1$, \mathbf{P}_e broadcasts \top otherwise it broadcasts \bot . The algorithm 10 gives the details of this procedure. This test returns \top if N is composite of two prime numbers but have a probability of 1/2 to return \top otherwise. Therefore, in order to increase

the reliability of this algorithm, the parties should run it s time with s such that $\frac{1}{2^n}$ is a probability low enough to return \top for a wrong modulus.

Algorithm 10 First discriminating biprimalty test

Parties elect a party \mathbf{P}_e \mathbf{P}_s generates $\gamma \in \mathbb{Z}_N^{\times}$ and shares with everybody for party \mathbf{P}_i (including \mathbf{P}_e) \mathbf{P}_i computes $\gamma_i = \gamma^{-p_i - q_i}$ \mathbf{P}_i sends γ_i to \mathbf{P}_e \mathbf{P}_e compute $\gamma_{final} = \gamma^{N+1} \cdot \prod_{i=1}^n \gamma_i \mod N$ if $|\gamma_{final}| = 1$, \mathbf{P}_e broadcasts \top else \bot

We can see it is important for γ to be in \mathbb{Z}_N^{\times} in order to be able to compute negative exponents by using γ^{-1} . The way of electing \mathbf{P}_e . A way that can be done is using the hashing function and the seed defined in the fast trial divisions in order to elect implicitly the leading party at each trial without requiring further communications.

Second part

In this part, we want compute $\gcd(\sum_{i\in\mathcal{N}}r_i\cdot(\sum_{i\in\mathcal{N}}p_i+\sum_{i\in\mathcal{N}}q_i-1),N)$ where r_i is a number generated by a node $i\in\mathcal{N}$. Let $\Delta_i=p_i+q_i, i\in\mathcal{N}$. In the same idea as for the distributed multiplication, we can rewrite $\sum_i\in\mathcal{N}r_i\cdot(\sum_i\in\mathcal{N}p_i+\sum_i\in\mathcal{N}q_i-1)$ as

$$\sum_{i \in \mathcal{N}} r_i \cdot \left(\sum_{i \in \mathcal{N}} p_i + \sum_{i \in \mathcal{N}} q_i - 1\right) = \sum_{i \in \mathcal{N}} (r_i \cdot \Delta_i - 1) + \sum_{i \in \mathcal{N}} \sum_{\substack{i \in \mathcal{N} \\ i \neq j}} r_i \cdot \Delta_i$$
 (3)

The have the same schema as for the distributed multiplication whith the first term of the left hand of Equation 3 which can be computed solely by a node and the other term has to be computed using distr_product. Finally they share all the sum g_i of their parts of computation similarly to f_i in the distributed multiplication and they compute $G = \sum_{i \in \mathcal{N}} g_i \mod N$. Finally each node can verify on its own wether $\gcd(G, N) = 1$. If so, the modulus N can be used, otherwise, the protocol has to be restarted.

Algorithm 11 Second discriminating biprimality test

```
g_{i} \leftarrow 0 \text{ for } i \in \overline{\mathcal{N}} Each party i generate a secret random number r_{i} \in \mathbb{Z}_{N} for i \in \mathcal{N} for 1 \leq i \leq j \leq n if i = j s_{i} += r_{i}(\Delta_{i} - 1) else \text{We compute } r_{i}\Delta_{j} \text{ and } r_{j}\Delta_{i} \text{ as in algorithm 9} g_{i} += u_{i}^{j,r} + u_{i}^{j,\Delta} g_{j} += u_{j}^{i,\Delta} + \beta_{j}^{i,r} The parties share g_{i} \mod N \ \forall i \in \mathcal{N} \text{ to compute } G = \sum_{i=1}^{n} g_{i} \mod N
```

5 Network overhead

Here we present a theoretical analysis of the amount of communications necessary for a successful try. We use **br** as an abbreviation for "broadcast".

if gcd(G, N) = 1 then N is considered to be sure

5.1 Fast trial divisions

Considering a single division, if we consider for simplicity we have $n=2^t$ participants, we have t steps where for the i-th step, 2^{t-i} parties send their values to someone else. In the worst case, which is the node which will return the value, it makes $\log_2(n)$ communications to receive the data plus one communication to return the result. In the best case, which is a node communicating it's private share modulus the small prime at the first round, it only makes on connection to give and one to receive the output.

So we have for B different fast trial divisions, each party operates the following amount of communications:

• Worst-Case : $B \cdot (\log_2(n) + 1)$

• Best-Case: $2 \cdot B$

5.2 Distributed multiplication

Let us count for a single multiplication of $a \cdot b$ with two k-bits numbers. For each bit the two parties instantiate an OT and either input or receives a value from it which makes k communications per party. Then as each party computes two product with the other parties, they operate 2k(n-1) communications plus 2k(n-1) OT instantiations. Finally we have a question of how sharing the final values in order to add everything to get N. We can suppose everyone broadcasts their final value f_i .

Finally we get 2k(n-1) + n communications per party for the distributed multiplication plus 2k(n-1) oblivious transfer initializations.

5.3 Biprimality test - First part

In this part, the chosen party to generate γ only broadcasts the value of γ , then the other parties communicate their exponentiation with a single communication and finally the first party communicates the output with a broadcast.

Then we have for each party according to the role during the s trials

• Worst-Case : $s \cdot (1 + (n-1) + 1) = s \cdot (n+1)$

• Best-Case : $3 \cdot s$

5.4 Biprimality test - Second part

Here we have something really similar to the distributed multiplication part except the multiplications ar 2k bits longs here. Then each party operates 4k(n-1) + n communications plus 4k(n-1) OT instantiations.

We can see the theoretical amount of communication is linear with the growth of n and k so we have a quadratic growth of the global amount of communications. Furthermore, we can parallelize a great part of the steps. For example we can run various fast trial divisions in parallel with a different prime number and massively parallelize the multiplications.

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1-out-of-n Oblivious Transfer

A 1-out-of-n Oblivious Transfer (OT) is a cryptosystem which allows a sender to propose n messages and a receiver to chose one of them without the sender knowing which one. We can also consider a random 1-out-of-n OT which sends the n messages to the sender instead of receiving an input. This cryptosystem allows two people to share information without disclosing knowledge on the choice of the receiver and the other options he could have chosen. This will be used as a way to compute distributed multiplications and in other situations in the protocol. The behaviour of this mechanism is specified in algorithm 12.

Algorithm 12 1-out-of-*n* Oblivious Transfer (OT) procedure

 ${f P1}$ and ${f P2}$ initialize the OT with the parameter n

P1 sends $\{\mu_1,\ldots,\mu_n\}$

P2 sends $i \in [1, n]$

P2 receives μ_i